NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2497

GENERALIZED CONICAL-FLOW FIELDS IN

SUPERSONIC WING THEORY

By Harvard Lomax and Max. A. Heaslet

Ames Aeronautical Laboratory
Moffett Field, Calif.

Washington
September 1951

TABLE OF CONTENTS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2497

GENERALIZED CONICAL-FLOW FIELDS IN SUPERSONIC WING THEORY

By Harvard Lomax and Max. A. Heaslet

SUMMARY

Linearized, compressible-flow analysis is applied to the study of quasi-conical supersonic wing theory. Single-integral equations are derived which relate either the loading to the shape of a lifting surface or the thickness of a symmetrical wing to the pressure distribution for triangular wings with subsonic leading edges. The forms of these equations and their inversions are simplified through the introduction of the finite part and the generalized principal part of an integral.

Applications of the theory, in the lifting case, include previously known results. In the nonlifting case, it is shown that for a specified pressure distribution the theory does not always predict a unique thickness distribution. This is demonstrated for a triangular plan form having a constant pressure gradient in the stream direction.

INTRODUCTION

If a sufficiently thin wing at a small angle of attack is placed in a uniform stream, its aerodynamic properties can be determined by means of the analysis associated with linearized compressible-flow theory. If, moreover, a Cartesian coordinate system is used such that the wing is situated on or in the immediate vicinity of the xy plane and the stream flows parallel to and in the direction of the positive x axis, it follows that the basic equation for the perturbation potential $\varphi(x,y,z)$ can be written in the form

$$\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \tag{1}$$

where $\beta^2 = M_0{}^2-1$, M_0 being the free-stream Mach number.

The application of equation (1) to wing theory is essentially a mathematical problem involving the solution of a differential equation with given boundary conditions. Consistent with the assumptions of linearized compressible flow theory, or small-perturbation theory, the boundary conditions expressing the prescribed physical conditions are given always at $z = 0$ and, as a consequence, boundary conditions as well as the solutions are superposable.

The techniques used in the solution of wing problems are, for the most part, adaptations of existing mathematical methods to the specific type of boundary values and their supporting surfaces that occur in aerodynamics. In particular, it is often possible in theory to make a reduction in the number of independent variables by virtue of known geometric or physical conditions. The conical-flow-field analysis of Busemann (reference 1) provides in this way a means of descending from a three- to a two-dimensional potential equation.

A conical flow field is one in which the perturbation velocity components and the induced pressures are constant in magnitude along any ray from the apex of the field. In this case, the perturbation potential may be written in the form

$$\varphi(x,y,z) = xf_1\left(\frac{\beta y}{x},\frac{\beta z}{x}\right) = \beta y f_2\left(\frac{x}{\beta y},\frac{\beta z}{\beta y}\right) = \beta z f_3\left(\frac{x}{\beta z},\frac{\beta y}{\beta z}\right) \qquad (2)$$

where φ is a homogeneous function of degree one in the three variables. An obvious generalization of this concept leads to the consideration of homogeneous potential fields of higher degree or, as they are sometimes called, quasi-conical fields. If φ is homogeneous of degree $\kappa + 1$, it follows that

$$\varphi(x,y,z) = x^{\kappa+1}\, F_1\left(\frac{\beta y}{x},\frac{\beta z}{x}\right) = (\beta y)^{\kappa+1}\, F_2\left(\frac{x}{\beta y},\frac{\beta z}{\beta y}\right) = (\beta z)^{\kappa+1}\, F_3\left(\frac{x}{\beta z},\frac{\beta y}{\beta z}\right) \qquad (3)$$

Equation (2) yields conical velocity fields, the degree of homogeneity being zero, while for equation (3) the quasi-conical velocity fields are homogeneous and of degree κ. Applications of these quasi-conical fields to pitching and rolling triangular wings have been given by Brown and Adams (reference 2), while Ribner (reference 3) has used similar methods in the consideration of cancellation elements. Further examples may be found in the literature.

If new variables are introduced in equation (1) such that

$$\frac{\beta y}{x} = \eta, \quad \frac{\beta z}{x} = \zeta, \quad \varphi(x,y,z) = cx^{\kappa+1}\,\Omega(\eta,\zeta) \qquad (4)$$

where c is an arbitrary constant, the transformed partial differential equation is

$$(\eta^2-1)\,\Omega_{\eta\eta} + 2\eta\zeta\,\Omega_{\eta\zeta} + (\zeta^2-1)\,\Omega_{\zeta\zeta} - \kappa[2\eta\,\Omega_\eta + 2\zeta\,\Omega_\zeta - (\kappa+1)\,\Omega] = 0 \quad (5)$$

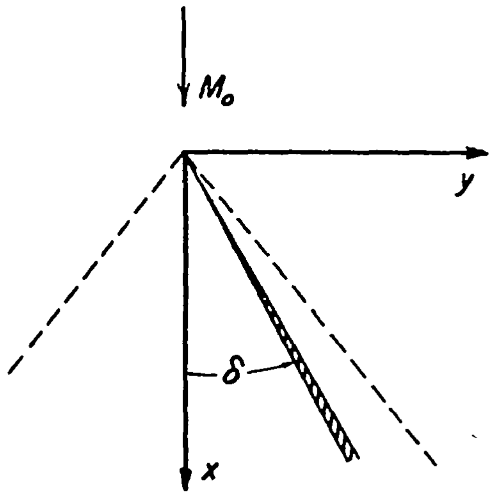
Thus, for a quasi-conical flow field with apex at the origin of the coordinate system, the resultant differential equation is elliptic for all values of η,ζ satisfying the inequality

$$\eta^2\zeta^2 - (1-\eta^2)(1-\zeta^2) < 0$$

that is, for all points inside the foremost Mach cone

$$x^2 - \beta^2y^2 - \beta^2z^2 = 0$$

The analysis of particular problems is therefore intimately associated with the study of two-dimensional, elliptic-type equations and is especially suited to the use of complex-variable theory. This is the approach taken by many investigators. In references 4 and 5, Lagerstrom and Germain have developed these methods in considerable detail.



A different approach to the study of lifting surfaces in conical flow fields has been given by Brown (reference 6) and in reference 7. In this approach a basic lifting element carrying a uniform load distribution and extending radially from the apex of the field (see sketch) is considered first. The induced velocity field is calculated for such an element lying in the plane of the wing and inclined to the stream direction at an arbitrary angle δ. The solution of a particular problem then proceeds along one of two lines. If the loading is given, the strength of each element is fixed and the calculation of the lifting-surface geometry depends only upon carrying out the integration. This is referred to as a direct problem. On the other hand, if the geometry of the wing is given, the loading is unknown and the strength of the elements must be adjusted so that the resultant vertical induced velocities are consistent with the given wing slope everywhere on the plan form. The solution of such a problem depends upon the inversion of a relatively simple singular integral equation and is referred to as an inverse problem.

As was pointed out in reference 8, similar methods apply to nonlifting problems in conical flow fields and pressure distributions corresponding to conical elements of thickness can be calculated. In such cases, however, the direct problem, that is, the one involving the evaluation of an integral, is the one in which the slope of the wing surface is given; and the inverse problem, that is, the one involving the solution of an integral equation is the one in which the shape of the pressure distribution is prescribed.

The present paper is concerned with the generalization of the basic elements of references 7 and 8 and their application to lifting and nonlifting problems in quasi-conical fields of flow for cases involving subsonic-type leading edges. Only solutions to the inverse problems will be considered and at all times these will be obtained by inverting the integral equation.

The orders of singularities that arise in the analysis are such that it is convenient to use the concepts of the finite part and generalized principal part of improper integrals. These generalizations will prove to be of importance for their notational efficiency and permit a simplified treatment of the derivatives of singular integrals. Hadamard (reference 9) has pointed out clearly the necessary steps in the treatment of such improper integrals, but did not stress the role of the differential operation in obtaining his integrals. Since some differences exist between Hadamard's definition of the finite part and the one used here, when extensions to multiple integrals appear, a different notation, consistent with reference 10 has been adopted. The generalization of the principal part has also been discussed in reference 3.

LIST OF IMPORTANT SYMBOLS

a_o	speed of sound in free stream
$C(\theta)$	load distribution on lifting surface as a function of θ
m	slope of radial element relative to free-stream direction
m_o	slope of right leading edge relative to free-stream direction
m_1	slope of left leading edge relative to free-stream direction
M_o	free-stream Mach number $\left(\frac{V_o}{a_o}\right)$
p	local static pressure
p_o	free-stream static pressure
P	angular rate of roll in radians per second
q	free-stream dynamic pressure $\left(\frac{1}{2}\rho_o V_o^2\right)$
Q	angular rate of pitch about wing vertex in radians per second

V_o	free-stream velocity
u	streamwise perturbation velocity
w	perturbation velocity normal to plane of wing
x,y,z	Cartesian coordinates introduced in equation (1)
α	wing angle of attack
β	$\sqrt{M_o^2 - 1}$
δ	angle between free-stream direction and line through wing vertex
δ_o	angle between right leading edge and stream direction
δ_1	angle between left leading edge and stream direction
Δu	discontinuity in u in plane of wing $(u_u - u_l)$
Δw	discontinuity in w in plane of wing $(w_u - w_l)$
$\Delta p/q$	load coefficient $\dfrac{(p_l - p_u)}{q}$
κ	constant determining degree of homogeneity of quasi-conical velocity field (See equation (3).)
ζ,η	conical variables introduced in equation (4) $\left(\frac{\beta z}{x}, \frac{\beta y}{x}\right)$
θ,θ_o,θ_1	$m\beta$, $m_o\beta$, $m_1\beta$
λ	slope of wing surface relative to free-stream direction
μ	Mach angle (arc cot β)
ρ_o	free-stream density
τ	region of integration in equations (10) and (27)
$\varphi(x,y,z)$	perturbation velocity potential introduced in equation (1)
$\Omega(\eta,\zeta)$	function related in equation (4) to perturbation velocity potential of a quasi-conical flow field

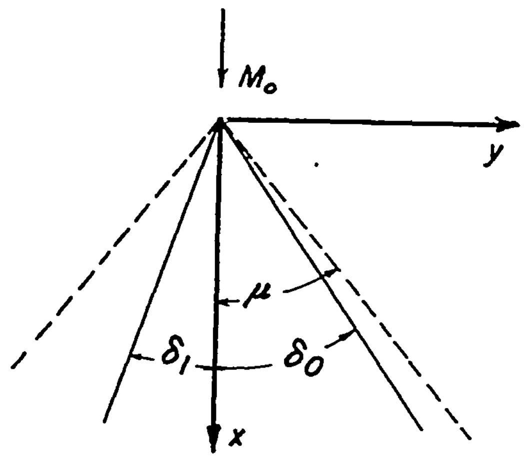
Subscripts

u denotes conditions on upper surface of wing

l denotes conditions on lower surface of wing

ANALYSIS

This investigation is confined to a consideration of inverse problems; that is, problems that require the inversion of an integral equation. As has been pointed out, these problems correspond to the two following cases: either the load distribution over a given lifting surface is to be determined or the thickness distribution corresponding to a prescribed pressure distribution is to be calculated. The given conditions must, of course, be such that a quasi-conical flow results. First, therefore, the plan form will be chosen, as shown in the accompanying sketch, so as to have an apex at the origin of coordinates and to be of semi-infinite extent. The traces of the foremost Mach cone are inclined to the positive x axis at the Mach angle $\pm\mu = \pm \text{arc cot } \beta$ and, since only subsonic leading edges are being considered, the leading edges of the plan form are inclined at angles smaller in magnitude than μ. Denoting these angles by δ_o and δ_1 and measuring them from the x axis positively in the conventional counterclockwise direction, it follows that the equations of the leading edges are



$$y = x \tan \delta_o = m_o x \quad \text{and} \quad y = x \tan \delta_1 = m_1 x \tag{6}$$

In the sketch δ_o is positive while δ_1 is negative.

The boundary conditions for the two types of problems may be stated as follows, where subscripts u and l are used to denote conditions at $z = 0+$ and $z = 0-$, respectively:

<u>Lifting case</u>: Over all the xy plane $\Delta w = w_u - w_l = 0$ and, except for the region occupied by the plan form, $\Delta u = u_u - u_l = 0$. On the plan form, vertical induced velocity is specified in either of the forms

$$w_u = w_l = V_o\, y^k\, \gamma_1 \left(\frac{x}{y}\right) = V_o\, x^k\, \gamma_2 \left(\frac{x}{y}\right)$$

where γ_1 or γ_2 are known polynomials in x/y.

Nonlifting case: Over all the xy plane $\Delta u = u_u - u_l = 0$ and, except for the region occupied by the plan form, $\Delta w = w_u - w_l = 0$. On the plan form, the streamwise induced velocity is specified in either of the forms

$$u_u = u_l = V_o\, y^k\, \nu_1 \left(\frac{x}{y}\right) = V_o\, x^k\, \nu_2 \left(\frac{x}{y}\right)$$

where ν_1 or ν_2 are known polynomials in x/y.

The solution to the two problems will be attained after considering first a lifting element and a thickness element and then, for each of these problems, the basic integral equation is determined by summing the appropriate elements. These derivations are given in the following sections. In small-perturbation theory the local load in coefficient form is related to u by the expressions

$$\frac{\Delta p}{q} = \frac{p_l - p_u}{\frac{1}{2}\rho_o {V_o}^2} = \frac{2\Delta u}{V_o} = \frac{4u_u}{V_o} \tag{7}$$

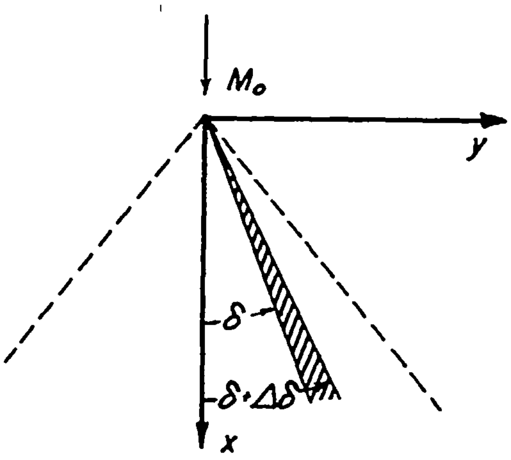
and the slope in the streamwise direction of an arbitrary surface $z = z(x,y)$ is related to vertical induced velocity by the expressions

$$\left.\begin{aligned} \lambda_u &= \frac{\partial z_u}{\partial x} = \frac{w_u}{V_o} \\ \lambda_l &= \frac{\partial z_l}{\partial x} = \frac{w_l}{V_o} \end{aligned}\right\} \tag{8}$$

Lifting Case

Upwash field of lifting element.— Consider a radial element emanating from the origin and assume that the load carried by the element is

$$\frac{\Delta p}{q} = C\ y^k \tag{9}$$



where C is a constant for a fixed position of the element. If the element is inclined to the x axis at an angle δ, its upwash field can be calculated by subtracting the induced fields of two triangular plan forms with vertex angles equal to $\delta + \Delta\delta$ and δ, each triangle having one side fixed for convenience along the x axis (see sketch). Assume, first, that δ is positive. As shown in reference 10, the upwash field of the triangle can be calculated from the known load distribution by means of the fundamental formula

$$\frac{w_u}{V_o} = \frac{1}{4\pi} \not\!\!\int_\tau dy_1 \int \frac{(x-x_1)\frac{\Delta p}{q}(x_1,y_1)\ dx_1}{(y-y_1)^2\ \sqrt{(x-x_1)^2-\beta^2(y-y_1)^2}} \tag{10}$$

where the region τ is the area on the plan form that lies ahead of the traces of the Mach forecone from the point (x,y,0). The bars on the integral sign indicate that the generalized principal part of the integral is to be evaluated (see appendix). By definition, if

$$\int \frac{F(y,y_1)dy_1}{(y_1-y)^2} = G(y,y_1) + \text{constant} \tag{11a}$$

is a known indefinite integral, the definite integral is evaluated as follows

$$\not\!\!\int_a^b \frac{F(y,y_1)dy_1}{(y_1-y)^2} = G(y,b) - G(y,a), \qquad a,b \neq y \tag{11b}$$

In case the singularity in the integrand lies outside the region of integration, the definition yields, of course, the conventional definite integral. In some cases in the following analysis the principal-part sign will be used to express relations valid for singularities both inside and outside the range of integration.

The integration with respect to x_1 in equation (10) is carried out over the area bounded by the lines

$$y_1 = 0, \quad x_1 = y_1 \cot \delta = y_1/m, \quad (x-x_1) = \pm\beta(y-y_1)$$

and, with the substitutions,

$$\theta = \beta m, \quad \eta = \beta y/x, \quad \eta_1 = \beta y_1/x \tag{12}$$

leads to the results

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \frac{\beta C}{4\pi} \int_0^{\frac{\theta(1+\eta)}{1+\theta}} \frac{{\eta_1}^{\kappa}}{(\eta-\eta_1)^2} \sqrt{\left(1 - \frac{\eta_1}{\theta}\right)^2 - (\eta-\eta_1)^2}\, d\eta_1; \quad 0<\theta,\ -1<\eta<\theta \tag{13a}$$

$$= \frac{\beta C}{4\pi} \int_0^{\frac{\theta(1-\eta)}{1-\theta}} \frac{{\eta_1}^{\kappa}}{(\eta-\eta_1)^2} \sqrt{\left(1 - \frac{\eta_1}{\theta}\right)^2 - (\eta-\eta_1)^2}\, d\eta_1; \quad 0<\theta,\ \theta<\eta<1 \tag{13b}$$

If θ and m are negative, the limits in equations (13a) and (13b) are reversed so that as the limits are now written changes in sign are required in the equations.

In order to obtain the upwash field of the required lifting element, it is sufficient to perform a direct differentiation for, if w_u of a plan form with vertex angle δ is of the form

$$w_u = f\,(\theta,\eta)$$

it follows that when $0<\theta$, the value of upwash induced by the element may be denoted dw_u and is

$$dw_u = f(\theta+\Delta\theta,\eta) - f(\theta,\eta) = \frac{\partial f}{\partial \theta}\, d\theta$$

and when $\theta<0$

$$dw_u = f(\theta,\eta) - f(\theta+\Delta\theta,\eta) = -\frac{\partial f}{\partial \theta}\, d\theta$$

If this process is carried out in equations (13a) and (13b) and if the transformation

$$\eta_1 = \theta\,\frac{\eta - t}{\theta - t}$$

is introduced, the expressions for dw_u become

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{dw_u}{V_o} = \frac{\beta C\ d\theta}{4\pi}\ \theta^{\kappa} \int_{-1}^{\eta} \frac{(\eta-t)^{\kappa+1}\ dt}{(\theta-\eta)(\theta-t)^{\kappa+1}\ t^2\sqrt{1-t^2}},\ -1<\eta<\theta<1 \qquad (14a)$$

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{dw_u}{V_o} = \frac{\beta C\ d\theta}{4\pi}\ \theta^{\kappa} \int_{1}^{\eta} \frac{(\eta-t)^{\kappa+1}\ dt}{(\theta-\eta)(\theta-t)^{\kappa+1}\ t^2\sqrt{1-t^2}},\ -1<\theta<\eta<1 \qquad (14b)$$

Equations (14a) and (14b) provide the upwash fields for any radial element, regardless of the sign of θ. Integration by parts leads to the alternative forms

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{dw_u}{V_o} = -\frac{\beta C\ d\theta(\kappa+1)\ \theta^{\kappa}}{4\pi} \int_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}\ dt}{(\theta-t)^{\kappa+2}\ t},\ -1<\eta<\theta<1 \qquad (15a)$$

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{dw_u}{V_o} = -\frac{\beta C\ d\theta(\kappa+1)\ \theta^{\kappa}}{4\pi} \int_{1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}\ dt}{(\theta-t)^{\kappa+2}\ t},\ -1<\theta<\eta<1 \qquad (15b)$$

where in one of the integrals the singularity in t requires the use of a Cauchy principal part.

Derivation and inversion of integral equation.- If now the lifting elements cover the region between θ_1 and θ_o and C is a function of θ determining the lift carried along the radial element at that point, upwash produced by the resultant plan form is given by the relation

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \lim_{\epsilon \to 0} -\frac{\beta(\kappa+1)}{4\pi}\left[\int_{\theta_1}^{\eta-\epsilon} \theta^{\kappa}\ C(\theta)\ d\theta \int_{1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}\ dt}{(\theta-t)^{\kappa+2}\ t} + \int_{\eta+\epsilon}^{\theta_o} \theta^{\kappa}\ C(\theta)\ d\theta \int_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{(\theta-t)^{\kappa+2}\ t}\ dt\right] \qquad (16a)$$

This result can be written as

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = -\frac{\beta(\kappa+1)}{4\pi}\int_{\theta_1}^{\theta_o} \theta^{\kappa}\ C(\theta)\ H(\theta,\eta)\ d\theta \qquad (16b)$$

where

$$H(\theta,\eta) = \int_1^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{(\theta-t)^{\kappa+2}\ t}\,dt, \qquad \theta_1 \leqq \theta < \eta$$

$$= \int_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{(\theta-t)^{\kappa+2}\ t}\,dt, \qquad {}_{o}\eta < \theta \leqq \theta_o$$

The function $H(\theta,\eta)$ has a simple pole at $\theta = \eta$, and the integral expression for w_u in equation (16b) is therefore evaluated as a Cauchy principal part.

The boundary condition to be satisfied by equations (16a) and (16b) is that $\beta^{\kappa}\ w_u/x^{\kappa}\ V_o$ is a polynomial of degree κ in the variable η. It follows that the $(\kappa+1)$ derivative of the right-hand member of equation (16b) must vanish.

Thus

$$0 = \frac{\beta(\kappa+1)}{4\pi} \neq\!\!\!\!\!\!\int_{\theta_1}^{\theta_o} \theta^{\kappa}\ C(\theta)\left(\frac{\partial}{\partial\eta}\right)^{\kappa+1} H(\theta,\eta)\ d\theta \tag{17}$$

where use is made of the generalized principal part of an integral defined as (see appendix)

$$\neq\!\!\!\!\!\!\int_a^b \frac{A(x_1)\ dx_1}{(x_1-x)^{n+1}} = \frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n \int_a^b \frac{A(x_1)\ dx_1}{x_1-x}$$

$$= -\frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^{n+1} \int_a^b A(x_1)\ln|x-x_1|dx_1$$

Here again the definition applies regardless of the value of x but is of particular significance when x lies within the region of integration.

Continuing the calculation in equation (17), one has, after taking the derivatives with respect to η,

$$0 = \frac{\sqrt{1-\eta^2}}{\eta} \neq\!\!\!\!\!\!\int_{\theta_1}^{\theta_o} \frac{\theta^{\kappa}\ C(\theta)\ d\theta}{(\theta-\eta)^{\kappa+2}}$$

which may be written in the form

$$\left(\frac{\partial}{\partial\eta}\right)^{\kappa+1} \int_{\theta_1}^{\theta_o} \frac{\theta^{\kappa}\, C(\theta)\, d\theta}{\theta-\eta} = 0$$

The function $C(\theta)$ is thus to be found through the inversion of the integral equation

$$\int_{\theta_1}^{\theta_o} \frac{\theta^{\kappa}\, C(\theta)\, d\theta}{\theta-\eta} = \sum_{i=o}^{\kappa} a_i\, \eta^i \tag{18}$$

The inversion of the integral equation

$$f(x) = \int_a^b \frac{g(\xi)\, d\xi}{x-\xi}, \qquad a<x<b$$

is known to be

$$g(x) = \frac{1}{\sqrt{(b-x)(x-a)}}\left[A - \int_a^b \frac{f(\xi)\sqrt{(b-\xi)(\xi-a)}}{x-\xi}\, d\xi\right] \tag{19}$$

where A is an arbitrary constant to be determined from physical considerations. Thus, the solution to equation (18) for $\theta_1<\theta<\theta_o$ is

$$\theta^{\kappa}\, C(\theta) = \frac{1}{\sqrt{(\theta_o-\theta)(\theta-\theta_1)}}\left[A - \int_{\theta_1}^{\theta_o} \frac{\sum_{i=o}^{\kappa} a_i\eta^i \sqrt{(\theta_o-\eta)(\eta-\theta_1)}}{\theta-\eta}\, d\eta\right]$$

and this leads to the expression

$$\theta^{\kappa}\, C(\theta) = \sum_{i=o}^{\kappa+1} \frac{b_i\theta^i}{\sqrt{(\theta_o-\theta)(\theta-\theta_1)}} \tag{20}$$

where the coefficients b_i are functions of θ_o and θ_1 but not functions of θ.

Relation of general solution to wing geometry.- From equation (9) the loading on the plan form is, since $\theta = \beta y/x$,

$$\frac{\Delta p}{q} = \left(\frac{x}{\beta}\right)^{\kappa} \sum_{i=0}^{\kappa+1} \frac{b_i \theta^i}{\sqrt{(\theta_0-\theta)(\theta-\theta_1)}} \tag{21}$$

where the coefficients b_i must be determined from known information about the surface geometry.

Consider next the identity

$$⨍_a^b \frac{dt}{(\theta-t)^{i+1}\sqrt{(b-t)(t-a)}} \equiv \frac{(-1)^i}{i!} \left(\frac{\partial}{\partial\theta}\right)^i \int_a^b \frac{dt}{(\theta-t)\sqrt{(b-t)(t-a)}} = 0,$$

$$a < \theta < b \tag{22}$$

where i is zero or a positive integer. This expression implies the equality

$$⨍_{-1}^{1} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{(\theta-t)^{\kappa+2}\ t} dt = 0$$

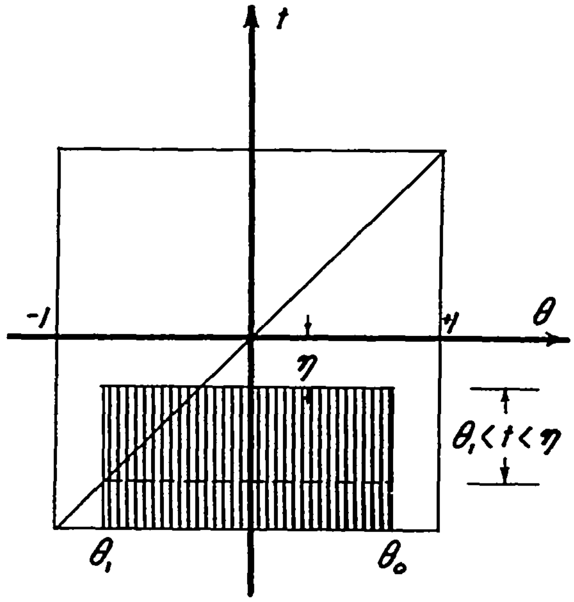
since the latter form can be broken, by expansion into rational fractions, into integrals like the left member of equation (22). The equality

$$⨍_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{(\theta-t)^{\kappa+2}\ t} dt = ⨍_{1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{(\theta-t)^{\kappa+2}\ t} dt$$

follows where the principal-part sign is needed on but one side of the equation, depending on the value of θ relative to η. From this result and equations (16) the following relations are obtained:

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_0} = -\frac{\beta(\kappa+1)}{4\pi} \int_{\theta_1}^{\theta_0} \theta^{\kappa}\ C(\theta)\ d\theta ⨍_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t(\theta-t)^{\kappa+2}} dt, \quad \eta < 0 \tag{23a}$$

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_0} = -\frac{\beta(\kappa+1)}{4\pi} \int_{\theta_1}^{\theta_0} \theta^{\kappa}\ C(\theta)\ d\theta ⨍_{1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t(\theta-t)^{\kappa+2}} dt, \quad \eta > 0 \tag{23b}$$



The range of η in these two equations has been restricted, respectively, to negative and positive values in order to avoid mathematical difficulties arising when singularities occur simultaneously at $t = \theta$ and $t = 0$. The shaded portion of the accompanying sketch is the region of integration in equation (23a). In the sketch, the inequalities $-1 \leqq \theta_1 < 0$, $0 < \theta_0 \leqq 1$ have been assumed. This implies that the plan form has two subsonic-type leading edges and such a condition will be assumed to apply henceforth.

It is particularly convenient to invert the order of integration in equations (23a) and (23b). Expressing the principal parts in the forms

$$-\!\!\!\!\!\!\int_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t(\theta-t)^{\kappa+2}}\,dt = \lim_{\alpha\to 0}\frac{1}{(\kappa+1)!}\left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}\int_{-1}^{\eta}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t(\theta-t-\alpha)}\,dt$$

$$-\!\!\!\!\!\!\int_{1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t(\theta-t)^{\kappa+2}}\,dt = \lim_{\alpha\to 0}\frac{1}{(\kappa+1)!}\left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}\int_{1}^{\eta}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t(\theta-t-\alpha)}\,dt$$

substituting from equation (20), and inverting the order of integration, leads one to the forms

$$\left(\frac{\beta}{x}\right)^{\kappa}\frac{w_u}{V_o} = \lim_{\alpha\to 0}\frac{-\beta}{4\pi\kappa!}\left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}\int_{-1}^{\eta}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\,dt \int_{\theta_1}^{\theta_o}\frac{\sum_{i=0}^{\kappa+1} b_i\theta^i\,d\theta}{(\theta-t-\alpha)\sqrt{(\theta_o-\theta)(\theta-\theta_1)}}, \qquad \eta<0$$

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \lim_{\alpha \to 0} \frac{-\beta}{4\pi\kappa!} \left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1} \int_{1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\, dt$$

$$\int_{\theta_1}^{\theta_o} \frac{\sum_{i=o}^{\kappa+1} b_i\, \theta^i\, d\theta}{(\theta-t-\alpha)\sqrt{(\theta_o-\theta)(\theta-\theta_1)}}, \qquad \eta > 0$$

Use of the algebraic identity

$$\frac{\theta^m}{(\theta-t-\alpha)} = \sum_{j=1}^{m} \theta^{m-j}\, (t+\alpha)^{j-1} + \frac{(t+\alpha)^m}{(\theta-t-\alpha)}$$

permits the expression of w_u in the alternative forms

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \lim_{\alpha \to 0} - \frac{\beta}{4\pi\kappa!} \left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}$$

$$\left[\int_{-1}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\, dt \int_{\theta_1}^{\theta_o} \frac{\sum_{i=o}^{\kappa+1} b_i \sum_{j=1}^{i} \theta^{i-j}(t+\alpha)^{j-1} d\theta}{\sqrt{(\theta_o-\theta)(\theta-\theta_1)}} + \right.$$

$$\int_{-1}^{\theta_1-\alpha} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}\, dt}{t} \int_{\theta_1}^{\theta_o} \frac{\sum_{i=o}^{\kappa+1} b_i\, (t+\alpha)^i\, d\theta}{(\theta-t-\alpha)\sqrt{(\theta_o-\theta)(\theta-\theta_1)}} +$$

$$\left. \int_{\theta_1-\alpha}^{\eta} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}\, dt}{t} \int_{\theta_1}^{\theta_o} \frac{\sum_{i=o}^{\kappa+1} b_i\, (t+\alpha)^i\, d\theta}{(\theta-t-\alpha)\sqrt{(\theta_o-\theta)(\theta-\theta_1)}} \right]$$

$$\left(\frac{\beta}{x}\right)^{\kappa}\frac{w_u}{V_o}=\lim_{\alpha\to o}-\frac{\beta}{4\pi\kappa!}\left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}$$

$$\left[\int_1^{\eta}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}dt}{t}\int_{\theta_1}^{\theta_o}\frac{\sum_{i=o}^{\kappa+1}b_i\sum_{j=1}^{i}\theta^{i-j}(t-\alpha)^{j-1}d\theta}{\sqrt{(\theta_o-\theta)(\theta-\theta_1)}}+\right.$$

$$\int_1^{\theta_o-\alpha}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}dt}{t}\int_{\theta_1}^{\theta_o}\frac{\sum_{i=o}^{\kappa+1}b_i\;(t+\alpha)^i\;d\theta}{(\theta-t-\alpha)\sqrt{(\theta_o-\theta)(\theta-\theta_1)}}+$$

$$\left.\int_{\theta_o-\alpha}^{\eta}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}dt}{t}\int_{\theta_1}^{\theta_o}\frac{\sum_{i=o}^{\kappa+1}b_i\;(t+\alpha)^i\;d\theta}{(\theta-t-\alpha)\sqrt{(\theta_o-\theta)(\theta-\theta_1)}}\right]$$

The double integrals occurring first in the right-hand members of these equations are of degree κ in α and their derivatives consequently vanish. Moreoever, from the identities

$$\int_a^b\frac{d\theta}{(\theta-x)\sqrt{(b-\theta)(\theta-a)}}=\begin{cases}\dfrac{-\pi}{\sqrt{(x-a)(x-b)}}, & b<x\\[2ex] 0, & a<x<b\\[2ex] \dfrac{\pi}{\sqrt{(b-x)(a-x)}}, & x<a\end{cases}\qquad(24)$$

it follows that the last terms in the right members vanish and that the expressions for w_u become

$$\left(\frac{\beta}{x}\right)^{\kappa}\frac{w_u}{V_o}=\lim_{\alpha\to o}\frac{-\beta}{4\kappa!}\left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}$$

$$\int_{-1}^{\theta_1-\alpha}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\sum_{i=o}^{\kappa+1}\frac{b_i(t+\alpha)^i}{\sqrt{(\theta_o-t-\alpha)(\theta_1-t-\alpha)}}dt,\;\eta<0$$

$$\left(\frac{\beta}{x}\right)^{\kappa}\frac{w_u}{V_o}=\lim_{\alpha\to 0}\frac{\beta}{4\kappa!}\left(\frac{\partial}{\partial\alpha}\right)^{\kappa+1}$$

$$\int_{1}^{\theta_o-\alpha}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\sum_{i=0}^{\kappa+1}\frac{b_i(t+\alpha)^i}{\sqrt{(t+\alpha-\theta_o)(t+\alpha-\theta_1)}}\,dt,\ \eta>0$$

If, as in the appendix, the finite part of an integral is

$$\int_a^b\frac{A(x_1)dx_1}{(x_1-b)^{i+1/2}}=\frac{2^i}{1.3\ldots(2i-1)}\int_a^b\left(\frac{\partial}{\partial b}\right)^i\frac{A(x_1)dx_1}{(x_1-b)^{1/2}}=$$

$$\frac{2^i}{1.3\ldots(2i-1)}\left(\frac{\partial}{\partial b}\right)^i\int_a^b\frac{A(x_1)dx_1}{(x_1-b)^{1/2}}$$

the expressions for w_u may be written in the form

$$\left(\frac{\beta}{x}\right)^{\kappa}\frac{w_u}{V_o}=\frac{-\beta}{4\kappa!}\int_{-1}^{\theta_1}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\left(\frac{\partial}{\partial t}\right)^{\kappa+1}\frac{\sum_{i=0}^{\kappa+1}b_it^i}{\sqrt{(\theta_o-t)(\theta_1-t)}}\,dt,\ \eta<0 \tag{25a}$$

$$\left(\frac{\beta}{x}\right)^{\kappa}\frac{w_u}{V_o}=\frac{\beta}{4\kappa!}\int_{1}^{\theta_o}\frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\left(\frac{\partial}{\partial t}\right)^{\kappa+1}\frac{\sum_{i=0}^{\kappa+1}b_it^i}{\sqrt{(t-\theta_o)(t-\theta_1)}}\,dt,\ \eta>0 \tag{25b}$$

Equations (25a) and (25b) are the fundamental equations for a lifting surface and, since w_u/V_o has been assumed known as a polynomial in $\eta = \beta y/x$, it remains merely to determine the unknown coefficients b_i by equating coefficients of η on both sides of the expressions. In the form given, it appears that for $\kappa > 0$ the number of equations obtainable exceeds the number of undetermined coefficients. No general theorems of determinancy have as yet been established as to the uniqueness of the solutions, but applications to be made later will indicate the techniques involved in calculating specific examples. In the laterally symmetrical case, where $-\theta_1=\theta_o$, solutions are easier to determine and the fundamental equations are

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \frac{-\beta}{4\kappa!} \int_{-1}^{-\theta_o} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t} \left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{i=o}^{\kappa+1} b_i t^i}{\sqrt{t^2-\theta_o^2}}\, dt, \quad \eta < 0 \tag{26a}$$

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \frac{\beta}{4\kappa!} \int_{1}^{\theta_o} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t} \left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{i=o}^{\kappa+1} b_i t^i}{\sqrt{t^2-\theta_o^2}}\, dt, \quad \eta > 0 \tag{26b}$$

Nonlifting Case

A radial element emanating from the origin is to be constructed such that it has a quasi-conical thickness distribution

$$\lambda_u = Cy^{\kappa}$$

where C is a constant and λ_u is the streamwise slope of the element as defined in equations (8). The derivation of the induced pressure field associated with the element follows closely the analysis in the lifting case. Thus, a triangular plan form is first considered where one side is parallel to the stream direction and with a vertex angle δ. From reference 10, pressure coefficient can be written in the form

$$C_p = -\frac{2}{\pi} \int_{\tau} \lambda_u \, dy_1 \int \frac{(x-x_1)\, dx_1}{[(x-x_1)^2-\beta^2(y-y_1)^2]^{3/2}} \tag{27}$$

where the region τ is the area on the plan form that lies ahead of the traces of the forecone from the point $(x, y, 0)$ and the integration with respect to x_1 involves the finite part.

In the notation of equations (13), the analogues to equations (15a) and (15b) are

$$\left(\frac{\beta}{x}\right)^{\kappa} dC_p = -\frac{2\,C\,d\theta\,\theta^{\kappa}(\kappa+1)}{\pi\beta} \int_{-1}^{\eta} \frac{t(\eta-t)^{\kappa}\,dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}, \quad -1 < \eta < \theta < 1 \tag{28a}$$

$$\left(\frac{\beta}{x}\right)^{\kappa} dC_p = -\frac{2\,C\,d\theta\,\theta^{\kappa}(\kappa+1)}{\pi\beta} \int_{1}^{\eta} \frac{t(\eta-t)^{\kappa}\,dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}, \quad -1 < \theta < \eta < 1 \tag{28b}$$

If C is a function of θ and the thickness elements cover the region between θ_1 and θ_o, pressure coefficient on the plan form is given by the expression

$$\left(\frac{\beta}{x}\right)^{\kappa} C_p = \lim_{\epsilon \to 0} \frac{-2(\kappa+1)}{\pi\beta} \left[\int_{\theta_1}^{\eta-\epsilon} \theta^{\kappa} C(\theta) d\theta \int_{1}^{\eta} \frac{t(\eta-t)^{\kappa}\, dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}} + \int_{\eta+\epsilon}^{\theta_o} \theta^{\kappa} C(\theta) d\theta \int_{-1}^{\eta} \frac{t(\eta-t)^{\kappa}\, dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}} \right] \tag{29}$$

The boundary conditions require $\beta^{\kappa} C_p / x^{\kappa}$ to be a polynomial of degree κ in η. If the $(\kappa+1)$st derivative of equation (29) is set equal to zero, the relation

$$0 = \int_{\theta_1}^{\eta} \theta^{\kappa} C(\theta) d\theta \left(\frac{\partial}{\partial\eta}\right)^{\kappa+1} \int_{1}^{\eta} \frac{t(\eta-t)^{\kappa}\, dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}} + \int_{\eta}^{\theta_o} \theta^{\kappa} C(\theta) d\theta \left(\frac{\partial}{\partial\eta}\right)^{\kappa+1} \int_{-1}^{\eta} \frac{t(\eta-t)^{\kappa}\, dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}$$

holds, and after further differentiation reduces to

$$0 = \int_{\theta_1}^{\theta_o} \frac{\theta^{\kappa} C(\theta) d\theta}{(\theta-\eta)^{\kappa+2}} = \left(\frac{\partial}{\partial\eta}\right)^{\kappa+1} \frac{1}{(\kappa+1)!} \int_{\theta_1}^{\theta_o} \frac{\theta^{\kappa} C(\theta) d\theta}{(\theta-\eta)}$$

The function $C(\theta)$ satisfies the same integral equation that arose in the lifting case. (See equation (18).) The solution can therefore be written, as in equation (21), in the form

$$\lambda_u = C(\theta)\, y^{\kappa} = \left(\frac{x}{\beta}\right)^{\kappa} \sum_{i=0}^{\kappa+1} \frac{a_i \theta^i}{\sqrt{(\theta_o-\theta)(\theta-\theta_1)}} \tag{30}$$

The equivalence

$$\int_{-1}^{\eta} \frac{t(\eta-t)^{\kappa}\, dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}} = \int_{1}^{\eta} \frac{t(\eta-t)^{\kappa}\, dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}$$

permits the rewriting of equation (29) in the forms

$$\left(\frac{\beta}{x}\right)^{\kappa} C_p = \frac{-2(\kappa+1)}{\pi\beta}\int_{\theta_1}^{\theta_0} C(\theta)\ \theta^{\kappa}\,d\theta \int_{1}^{\eta} \frac{t(\eta-t)^{\kappa}\ dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}\ ,\ \eta>0 \tag{31a}$$

$$\left(\frac{\beta}{x}\right)^{\kappa} C_p = \frac{-2(\kappa+1)}{\pi\beta}\int_{\theta_1}^{\theta_0} C(\theta)\ \theta^{\kappa}\,d\theta \int_{-1}^{\eta} \frac{t(\eta-t)^{\kappa}\ dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}\ ,\ \eta<0 \tag{31b}$$

Substitution for $\theta^{\kappa}C(\theta)$ from equation (30), inversion of the order of integration, and use of equations (24) leads to the fundamental relations

$$\left\{\begin{aligned}
\left(\frac{\beta}{x}\right)^{\kappa} C_p &= -\frac{2}{\beta\kappa!}\int_{-1}^{\theta_1} \frac{t(\eta-t)^{\kappa}}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{i=0}^{\kappa+1} a_i t^i}{\sqrt{(\theta_0-t)(\theta_1-t)}}\,dt,\ \eta<0 && (32a)\\
\left(\frac{\beta}{x}\right)^{\kappa} C_p &= \frac{2}{\beta\kappa!}\int_{1}^{\theta_0} \frac{t(\eta-t)^{\kappa}}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{i=0}^{\kappa+1} a_i t^i}{\sqrt{(t-\theta_0)(t-\theta_1)}}\,dt,\ \eta>0 && (32b)
\end{aligned}\right.$$

When $\theta_1 = -\theta_0$, these equations become

$$\left\{\begin{aligned}
\left(\frac{\beta}{x}\right)^{\kappa} C_p &= -\frac{2}{\beta\kappa!}\int_{-1}^{-\theta_0} \frac{t(\eta-t)^{\kappa}}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{i=0}^{\kappa+1} a_i t^i}{\sqrt{t^2-\theta_0{}^2}}\,dt && (33a)\\
\left(\frac{\beta}{x}\right)^{\kappa} C_p &= \frac{2}{\beta\kappa!}\int_{1}^{\theta_0} \frac{t(\eta-t)^{\kappa}}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{i=0}^{\kappa+1} a_i t^i}{\sqrt{t^2-\theta_0{}^2}}\,dt && (33b)
\end{aligned}\right.$$

The determination of the thickness distribution corresponding to a given pressure distribution can thus be obtained from the above equations by equating coefficients of η and solving for the unknown coefficients a_i. Specific examples will serve to make the steps clearer; such problems will be considered in the following section.

APPLICATIONS

Since homogeneous fields of low degree have already received considerable attention, several results have been published previously. In the case of thickness problems corresponding to specified pressure distributions, however, solutions have never, so far as is known, been sought in terms of the given pressures. Rather, the thickness has been assumed known and the resulting pressure distribution calculated. This latter attack involves no question as to uniqueness and a one-to-one correspondence certainly exists. When pressure is prescribed first, however, it becomes necessary to consider the possibility of nonuniqueness. In two-dimensional, low-speed flow a freedom of choice is known to exist and leads to the introduction of purely circulatory flow which, in turn, provides the mechanism of lift. No analogue to this occurs in the low-speed, two-dimensional, nonlifting case when the body is smooth and is assumed to close. In the following developments a multiplicity of solutions will, however, occur in the nonlifting case and bodies with given pressure distributions retain a degree of freedom.

Equations (25) and (26), together with equation (21), suffice for the solution of quasi-conical lifting problems while equations (32) and (33), together with equation (30), apply to symmetrical wings. In the applications to follow, the division into lifting and nonlifting cases has been maintained. The detailed treatment of equations (26) and (33) can be further simplified if the problems are separated into cases involving symmetry and antisymmetry about the x axis of the imposed boundary conditions. Suppose, first, that the given values of upwash and pressure coefficient in these equations are odd functions of η. It follows from physical considerations that the loading or surface slope, respectively, will be an odd function of η and that consequently the unknown coefficients a_i or b_i must vanish for even values of the subscript i. If the transformation $t = -\tau$ is made in either equation (26b) or (33b), the pairs of equations (26a) and (26b) or (33a) and (33b) yield consistent sets of simultaneous linear equations that can be obtained from single equalities. Hence, for w_u or C_p expressed in odd powers of η,

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \frac{+\beta}{4\kappa!}\int_1^{\theta_o} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t} \left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{j=0}^{[\kappa/2]} b_{2j+1}\, t^{2j+1}}{\sqrt{t^2-\theta_o{}^2}}\, dt \tag{34}$$

or

$$\left(\frac{\beta}{x}\right)^{\kappa} C_p = \frac{2}{\beta\kappa!}\int_1^{\theta_o} \frac{t(\eta-t)^{\kappa}}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{j=0}^{[\kappa/2]} a_{2j+1}\, t^{2j+1}}{\sqrt{t^2-\theta_o^2}}\, dt \tag{35}$$

where the notation $[\kappa/2]$ in the summation denotes the largest integer contained in $\kappa/2$.

In the same fashion, a simplification can be achieved for symmetrical boundary conditions. Again using the bracket notation to indicate the largest integer, the resulting equations become, when w_u or C_p are expressed in even powers of η,

$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_o} = \frac{+\beta}{4\kappa!}\int_1^{\theta_o} \frac{(\eta-t)^{\kappa}\sqrt{1-t^2}}{t}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{j=0}^{[(\kappa+1)/2]} b_{2j}\, t^{2j}}{\sqrt{t^2-\theta_o^2}}\, dt \tag{36}$$

or

$$\left(\frac{\beta}{x}\right)^{\kappa} C_p = \frac{2}{\beta\kappa!}\int_1^{\theta_o} \frac{t(\eta-t)^{\kappa}}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^{\kappa+1} \frac{\sum_{j=0}^{[(\kappa+1)/2]} a_{2j}\, t^{2j}}{\sqrt{t^2-\theta_o^2}}\, dt \tag{37}$$

Wings With Load Distributions

The yawed triangular wing.- Equations (21) and (25) lead directly to the determination of angle-of-attack loading on a yawed triangular wing. This solution is well known and was calculated in reference 7 by a method which was a particular case of the present theory.

The boundary conditions are that $w_u = -V_o\alpha$ on the plan form, hence $\kappa = 0$ and, from equation (21),

$$\frac{\Delta p}{q} = \frac{b_o+b_1\theta}{\sqrt{(\theta_o-\theta)(\theta-\theta_1)}} \tag{38}$$

Equations (25) lead to the equalities

$$-\alpha = \frac{\beta}{4}\int_1^{\theta_0} \frac{\sqrt{1-t^2}}{t} \frac{\partial}{\partial t} \frac{b_0+b_1 t}{\sqrt{(t-\theta_0)(t-\theta_1)}}\, dt$$

$$-\alpha = \frac{-\beta}{4}\int_{-1}^{\theta_1} \frac{\sqrt{1-t^2}}{t} \frac{\partial}{\partial t} \frac{b_0+b_1 t}{\sqrt{(\theta_0-t)(\theta_1-t)}}\, dt$$

After integration by parts, these relations become

$$\alpha = -\frac{\beta}{4}\int_1^{\theta_0} \frac{(b_0+b_1 t)\, dt}{t^2\sqrt{(1-t^2)(t-\theta_0)(t-\theta_1)}}$$

$$\alpha = \frac{\beta}{4}\int_{-1}^{\theta_1} \frac{(b_0+b_1 t)\, dt}{t^2\sqrt{(1-t^2)(\theta_0-t)(\theta_1-t)}}$$

Hence,

$$\left.\begin{aligned} \alpha &= \frac{\beta}{4}\left[b_0\, L_0(\theta_0, \theta_1) + b_1\, L_1(\theta_0, \theta_1) \right] \\ \alpha &= \frac{\beta}{4}\left[b_0\, L_0(-\theta_1, -\theta_0) - b_1\, L_1(-\theta_1, -\theta_0) \right] \end{aligned}\right\} \tag{39}$$

where

$$L_0(\theta_0, \theta_1) = -\int_1^{\theta_0} \frac{dt}{t^2\sqrt{(1-t^2)(t-\theta_0)(t-\theta_1)}}$$

and

$$L_1(\theta_0, \theta_1) = -\int_1^{\theta_0} \frac{dt}{t\sqrt{(1-t^2)(t-\theta_0)(t-\theta_1)}}$$

It is apparent that equations (39) can be solved for b_0 and b_1 in terms of the functions $L_0(\theta_0, \theta_1)$ and $L_1(\theta_0, \theta_1)$. Substitution of the values into equation (38) leads to the expression for load coefficient

$$\frac{\Delta p}{q} = \frac{2\alpha}{\beta E}\sqrt{\frac{2G}{\theta_0-\theta_1}}\left[\frac{(\theta_0+\theta_1)\,\theta-2\theta_0\theta_1}{\sqrt{(\theta_0-\theta)(\theta-\theta_1)}}\right] \tag{40}$$

where E is the complete elliptic integral of the second kind with modulus $\sqrt{1-G^2}$ and

$$G = \frac{1-\theta_0\theta_1 - \sqrt{(1-\theta_0^2)(1-\theta_1^2)}}{\theta_0-\theta_1} = \frac{\left[\sqrt{(1-\theta_0)(1+\theta_1)} - \sqrt{(1-\theta_1)(1+\theta_0)}\right]^2}{2(\theta_0-\theta_1)} \tag{41}$$

In the particular case when $\theta_1 = -\theta_0$, the value of G becomes θ_0 and the resultant loading on the unyawed wing becomes

$$\frac{\Delta p}{q} = \frac{4\alpha\,\theta_0^2}{\beta E\sqrt{\theta_0^2-\theta^2}}$$

where the modulus of E is $\sqrt{1-\theta_0^2}$. This latter problem could, of course, have been solved directly from equation (36) in a much simpler fashion.

The rolling triangular wing.- Consider next the case of an unyawed triangular wing rolling about its axis of symmetry. If the angular rate of roll is P radians per second, the boundary conditions on the wing are that $w_u = -Py$. In this case $\kappa = 1$ and, from equation (21), the loading on the wing is given by the expression

$$\frac{\Delta p}{q} = \frac{x}{\beta}\frac{b_1\theta}{\sqrt{\theta_0^2-\theta^2}} \tag{42}$$

where the coefficients b_0 and b_2 can be deleted since the loading must obviously be antisymmetrical. Since $[\kappa/2]$ is equal to zero in equation (34), b_1 is found to satisfy the relations

$$\frac{P}{V_0} = \frac{-\beta b_1}{4}\int_1^{\theta_0}\frac{\sqrt{1-t^2}}{t}\left(\frac{\partial}{\partial t}\right)^2\frac{t}{\sqrt{t^2-\theta_0^2}}\,dt \tag{43a}$$

$$0 = b_1\int_1^{\theta_0}\sqrt{1-t^2}\left(\frac{\partial}{\partial t}\right)^2\frac{t}{\sqrt{t^2-\theta_0^2}}\,dt \tag{43b}$$

Performing one differentiation with respect to t, in each of the two integrals, and then integrating by parts, leads one to the expressions

$$\frac{P}{V_0} = \frac{\beta b_1\theta_0^2}{4}\int_1^{\theta_0}\frac{dt}{t^2\sqrt{1-t^2}\,(t^2-\theta_0^2)^{3/2}} = \frac{\beta b_1\theta_0}{4}\frac{\partial}{\partial\theta_0}\int_1^{\theta_0}\frac{dt}{t^2\sqrt{(1-t^2)(t^2-\theta_0^2)}}$$

$$0 = b_1\theta_o^2 \oint_1^{\theta_o} \frac{t\,dt}{\sqrt{1-t^2}\,(t^2-\theta_o^2)^{3/2}} = b_1\theta_o \frac{\partial}{\partial\theta_o}\int_1^{\theta_o} \frac{t\,dt}{\sqrt{(1-t^2)(t^2-\theta_o^2)}}$$

The last equation reduces to an identity while the former one becomes

$$\frac{P}{V_o} = \frac{-\beta b_1\theta_o}{4}\frac{\partial}{\partial\theta_o}\left(\frac{E}{\theta_o^2}\right) = \frac{\beta b_1}{4\theta_o^2}\left[\frac{2E-\theta_o^2\,(E+K)}{(1-\theta_o^2)}\right] \tag{44}$$

where the modulus of the complete elliptic integrals E and K is $k = \sqrt{1-\theta_o^2}$. From equations (42) and (44) the loading is

$$\frac{\Delta p}{q} = \frac{4P\,\theta_o^2\,x\,\theta}{\beta^2 V_o\left(\frac{2-\theta_o^2}{1-\theta_o^2}E - \frac{\theta_o^2}{1-\theta_o^2}K\right)\sqrt{\theta_o^2-\theta^2}} \tag{45}$$

and is in agreement with the results of reference 2.

<u>The pitching triangular wing</u>.— If an unyawed triangular wing is pitching about its vertex, the boundary conditions become $w_u = -Qx$ where Q is the rate of pitch in radians per second. From equations (21) and (36), the loading and the relation involving the undetermined constants are

$$\frac{\Delta p}{q} = \frac{x}{\beta}\frac{b_o+b_2\theta^2}{\sqrt{\theta_o^2-\theta^2}}$$

and

$$\frac{-\beta Q}{V_o} = \frac{\beta}{4}\oint_1^{\theta_o} \frac{(\eta-t)\sqrt{1-t^2}}{t}\left(\frac{\partial}{\partial t}\right)^2 \frac{b_o+b_2t^2}{\sqrt{t^2-\theta_o^2}}\,dt$$

since $\kappa = 1$ and the loading must be symmetrical about the x axis. Equating coefficients of η in the latter equation, one has

$$0 = \oint_1^{\theta_o} \frac{\sqrt{1-t^2}}{t}\left(\frac{\partial}{\partial t}\right)^2 \frac{b_o+b_2t^2}{\sqrt{t^2-\theta_o^2}}\,dt \tag{46a}$$

$$\frac{Q}{V_o} = \frac{1}{4}\int_1^{\theta_o} \sqrt{1-t^2}\left(\frac{\partial}{\partial t}\right)^2 \frac{b_o+b_2t^2}{\sqrt{t^2-\theta_o^2}}\,dt \tag{46b}$$

or, after carrying out one differentiation and integrating by parts,

$$0 = b_2 \frac{\partial}{\partial\theta_o}\int_1^{\theta_o} \frac{t\,dt}{\sqrt{(1-t^2)(t^2-\theta_o^2)}} - (b_o+2b_2\theta_o^2)\frac{\partial}{\partial\theta_o}\int_1^{\theta_o} \frac{dt}{t\sqrt{(1-t^2)(t^2-\theta_o^2)}}$$

$$\frac{4Q}{V_o}\theta_o = -b_2\frac{\partial}{\partial\theta_o}\int_{\theta_o}^{1} \frac{t^4\,dt}{\sqrt{(1-t^2)(t^2-\theta_o^2)}} +$$

$$(b_o+2b_2\theta_o^2)\frac{\partial}{\partial\theta_o}\int_{\theta_o}^{1} \frac{t^2\,dt}{\sqrt{(1-t^2)(t^2-\theta_o^2)}}$$

The first term in the first equation is zero and, after integration of the terms in the second equation, the following relations are obtained

$$0 = b_o + 2\theta_o^2\,b_2$$

$$\frac{-4k^2Q}{V_o} = b_o(E-K) + (E-\theta_o^2K)b_2$$

where the modulus of the complete elliptic integrals is $k = \sqrt{1-\theta_o^2}$.

The solutions of these simultaneous equations are

$$b_o = \frac{+8\,\theta_o^2k^2Q}{V_o\left[\,\theta_o^2K+(1-2\theta_o^2)E\,\right]},\quad b_2 = \frac{-4\,k^2Q}{\left[\,\theta_o^2K+(1-2\theta_o^2)E\,\right]V_o} \tag{47}$$

and the resultant loading on the wing is

$$\frac{\Delta p}{q} = \frac{4x\,Q}{V_o\beta\sqrt{\theta_o^2-\theta^2}}\,\frac{2\,\theta_o^2-\theta^2}{\left(\frac{\theta_o^2}{1-\theta_o^2}K + \frac{1-2\theta_o^2}{1-\theta_o^2}E\right)} \tag{48}$$

Differentially deflected triangular wing.- If the two sides of an unyawed triangular wing are deflected differentially, vertical induced velocity on the wing is

$$\frac{w_u}{V_o} = \frac{-y}{|y|}\alpha$$

and loading must therefore be asymmetrical. It follows that $\kappa = 0$ and, from equations (21) and (34),

$$\frac{\Delta p}{q} = \frac{b_1 \theta}{\sqrt{\theta_o^2 - \theta^2}}$$

and

$$-\alpha = \frac{\beta}{4} \int_1^{\theta_o} \frac{\sqrt{1-t^2}}{t} \frac{\partial}{\partial t} \frac{b_1 t}{\sqrt{t^2 - \theta_o^2}} dt$$

An integration by parts, in the latter expression, reduces the integral to a standard form. A further integration leads to the equality

$$b_1 = \frac{8 \theta_o \alpha}{\beta \pi}$$

and load distribution is therefore

$$\frac{\Delta p}{q} = \frac{8\alpha \theta_o \theta}{\beta \pi \sqrt{\theta_o^2 - \theta^2}} \tag{49}$$

Triangular wing with parabolic twist.- Consider, finally, an unyawed triangular wing twisted symmetrically such that its vertical induced velocity is of the form

$$\frac{w_u}{V_o} = r y^2$$

where r is a fixed constant. Since $\kappa = 2$, the relations

$$\frac{\Delta p}{q} = \left(\frac{x}{\beta}\right)^2 \frac{b_o + b_2 \theta^2}{\sqrt{\theta_o^2 - \theta^2}}$$

and

$$r\eta^2 = \frac{\beta}{8} \int_1^{\theta_o} \frac{(\eta - t)^2 \sqrt{1-t^2}}{t} \left(\frac{\partial}{\partial t}\right)^3 \frac{b_o + b_2 t^2}{\sqrt{t^2 - \theta_o^2}} dt$$

apply. Detailed analysis will be omitted in this case since it follows the same pattern of development used in the earlier cases. Three simultaneous equations involving the two unknowns b_o and b_2 are obtained, but the equation relating coefficients of η can be shown to vanish identically and the two remaining equations yield a unique answer. The expression for load distribution is

$$\frac{\Delta p}{q} = \frac{8r}{\beta}\left(\frac{x}{\beta}\right)^2 \theta_o^4 \frac{\left[2\theta_o^2K-(1+\theta_o^2)E\right]+\left[-(3-\theta_o^2)K+4(2-\theta_o^2)E\right]\theta^2}{\left[-5\theta_o^4K^2+8\theta_o^2(1+\theta_o^2)KE+(4\theta_o^4-19\theta_o^2+4)E^2\right]\sqrt{\theta_o^2-\theta^2}} \tag{50}$$

where the modulus of the complete elliptic integrals K and E is $\sqrt{1-\theta_o^2}$.

Wings With Thickness Distributions

Triangular wing with uniform pressure.- In reference 11, Squire considered certain thickness distributions for symmetrical nonlifting wings in conical flow fields and calculated the resultant pressure distribution. The first of Squire's examples was a triangular plan form with a uniform pressure distribution. It is instructive to consider the inverse of this problem and to seek the wing assuming the pressure variation known. The plan form is symmetrically disposed with respect to the x axis or stream direction while the boundary conditions require that C_p is a constant over the entire wing. In this case, $\kappa = 0$ and from equations (30) and (37) the following relations hold

$$\frac{dz_u}{dx} = \lambda_u = \frac{a_o}{\sqrt{\theta_o^2-\theta^2}} \tag{51}$$

$$C_p = \frac{2}{\beta}\int_1^{\theta_o} \frac{t}{\sqrt{1-t^2}} \frac{\partial}{\partial t} \frac{a_o}{\sqrt{t^2-\theta_o^2}}\, dt \tag{52}$$

The value of a_o may be found in a manner quite similar to the one used in the previous examples. Thus, after the differentiation is performed, equation (52) becomes

$$\frac{\beta C_p}{2} = -a_o\int_1^{\theta_o} \frac{t^2\, dt}{\sqrt{1-t^2}\,(t^2-\theta_o^2)^{3/2}} = -\frac{a_o}{\theta_o}\frac{\partial}{\partial\theta_o}\int_1^{\theta_o} \frac{t^2\, dt}{\sqrt{(1-t^2)(t^2-\theta_o^2)}}$$

After integration, this yields

$$\frac{\beta C_p}{2} = \frac{a_o}{\theta_o}\frac{\partial}{\partial\theta_o}E = \frac{a_o(K-E)}{1-\theta_o^2} \tag{53}$$

where the modulus of the elliptic integrals is $\sqrt{1-\theta_o^2}$. If a_o is eliminated from equations (51) and (53), the slope of the wing on the upper surface is

$$\frac{dz_u}{dx} = \frac{\beta C_p(1-\theta_o^2)}{2(K-E)}\frac{1}{\sqrt{\theta_o^2-\theta^2}} = \frac{C_p(1-\theta_o^2)}{2(K-E)}\frac{x}{\sqrt{m^2x^2-y^2}}$$

The ordinate of the upper surface results from the integration

$$z_u = \int_{\frac{y}{m}}^{x}\frac{dz_u}{dx}\,dx$$

and is

$$z_u = \frac{(1-\theta_o^2)C_p}{2m^2(K-E)}\sqrt{m^2x^2-y^2} = \frac{(1-\theta_o^2)C_p x}{2\beta m^2(K-E)}\sqrt{\theta_o^2-\theta^2} \tag{54}$$

Triangular wing with linear pressure gradient.— It is now proposed to determine the thickness distribution for an unyawed triangular wing for which pressure varies linearly in the streamwise direction. Setting C_p equal to bx, it follows from equation (30) that the slope of the upper surface is expressible as

$$\frac{dz_u}{dx} = \frac{x}{\beta}\frac{a_o+a_2\theta^2}{\sqrt{\theta_o^2-\theta^2}} \tag{55}$$

since the solution is obviously symmetrical about the stream axis. Since $\kappa = 1$, equation (37) becomes

$$\beta b = \frac{2}{\beta}\int_1^{\theta_o}\frac{t(\eta-t)}{\sqrt{1-t^2}}\left(\frac{\partial}{\partial t}\right)^2\frac{a_o+a_2t^2}{\sqrt{t^2-\theta_o^2}}\,dt \tag{56}$$

Equation (56) is an identity and, after the coefficients of η are equated, leads to the two relations

$$0 = \int_1^{\theta_o} \frac{t}{\sqrt{1-t^2}} \left(\frac{\partial}{\partial t}\right)^2 \frac{a_o+a_2t^2}{\sqrt{t^2-\theta_o{}^2}}\, dt \tag{57a}$$

$$\frac{\beta^2 b}{2} = -\int_1^{\theta_o} \frac{t^2}{\sqrt{1-t^2}} \left(\frac{\partial}{\partial t}\right)^2 \frac{a_o+a_2t^2}{\sqrt{t^2-\theta_o{}^2}}\, dt \tag{57b}$$

It is of interest to compare the series of equations just developed with the corresponding equations in the problem of the pitching triangular wing. Formally, the algebraic steps are the same and it is to be expected that, just as in the case of equations (46), two simultaneous equations will be obtained and that their solutions will provide the constants a_o and a_2. In the present case, however, equation (57a) can be shown to vanish identically and as a result only one equation in two unknowns remains. This means that an infinite number of possible solutions exists. The following calculations will supply the necessary details to confirm these remarks.

Consider equation (57a) and introduce the transformation

$$t^2 = \tau, \quad \theta_o{}^2 = \tau_o \tag{58}$$

Then, by means of the relations

$$\frac{\partial}{\partial t} = 2\sqrt{\tau}\,\frac{\partial}{\partial \tau}, \quad \frac{\partial^2}{\partial t^2} = 2\left(2\tau\,\frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial \tau}\right) \tag{59}$$

the equation becomes

$$\begin{aligned}
0 &= a_o \int_1^{\tau_o} \frac{d\tau}{\sqrt{1-\tau}} \left(2\tau\,\frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial \tau}\right) \frac{1}{\sqrt{\tau-\tau_o}} + {} \\
&\quad a_2 \int_1^{\tau_o} \frac{d\tau}{\sqrt{1-\tau}} \left(2\tau\,\frac{\partial^2}{\partial \tau^2} + \frac{\partial}{\partial \tau}\right) \frac{\tau}{\sqrt{\tau-\tau_o}} \\
&= a_o\left[\frac{\partial^2}{\partial \tau_o{}^2}\int_1^{\tau_o} \frac{2\tau\, d\tau}{\sqrt{(1-\tau)(\tau-\tau_o)}} - \frac{\partial}{\partial \tau_o} \int_1^{\tau_o} \frac{d\tau}{\sqrt{(1-\tau)(\tau-\tau_o)}}\right] + {} \\
&\quad a_2\left[\frac{\partial^2}{\partial \tau_o{}^2}\int_1^{\tau_o} 2\tau\sqrt{\frac{\tau-\tau_o}{1-\tau}}\, d\tau - \frac{\partial}{\partial \tau_o}\int_1^{\tau_o}\sqrt{\frac{\tau-\tau_o}{1-\tau}}\, d\tau + {}\right. \\
&\quad \left. 2\tau_o\,\frac{\partial^2}{\partial \tau_o{}^2}\int_1^{\tau_o} \frac{\tau\, d\tau}{\sqrt{(1-\tau)(\tau-\tau_o)}} - \tau_o\,\frac{\partial}{\partial \tau_o}\int_1^{\tau_o} \frac{d\tau}{\sqrt{(1-\tau)(\tau-\tau_o)}}\right]
\end{aligned}$$

Each of the above integrals can be evaluated directly and the bracketed terms are in both cases zero. Since similar integrals occur in problems of this type, however, it is worthwhile to give the following general formulas (n an integer)

$$\int_1^{\tau_0} \frac{\tau^n \, d\tau}{\sqrt{(1-\tau)(\tau-\tau_0)}} = -\pi, \quad n = 0; \; = -\frac{\pi}{2}(1+\tau_0), \; n=1;$$

$$= -\frac{\pi}{2^{2n-1}} \left[\frac{(2n-1)!}{(n-1)!n!} \left(\tau_0^{\,n} + 1\right) + \sum_{j=1}^{n-1} \frac{2(2j-1)!(2n-2j-1)!}{(n-j)!j!(j-1)!(n-j-1)!} \tau_0^{\,n-j} \right], \; n>1 \tag{60a}$$

and

$$\int_1^{\tau_0} \tau^n \sqrt{\frac{\tau-\tau_0}{1-\tau}} \, d\tau = \frac{\tau_0-1}{2} \pi, \quad n = 0; \; = \frac{\pi}{8}\left(\tau_0^{\,2} + 2\tau_0 - 3\right), \; n=1;$$

$$= \frac{\pi(\tau_0-1)}{2^{2n}(n+1)} \left[\frac{(2n-1)!\tau_0^{\,n}}{(n-1)!n!} + \frac{(2n+1)!}{2\, n!\, n!} + \sum_{j=1}^{n-1} \frac{(2j+1)!(2n-2j-1)!}{(n-j)!j!j!(n-j-1)!} \tau_0^{\,n-j} \right], \; n>1 \tag{60b}$$

It remains to calculate the terms in equation (57b). If the differential relations in equation (59) are used, the desired expression is

$$\frac{-\beta^2 b}{2} = a_0 \left(2 \frac{\partial^2}{\partial \tau_0^{\,2}} D_1 + \frac{\partial}{\partial \tau_0} D_2 \right) + a_2 \left(2 \frac{\partial^2}{\partial \tau_0^{\,2}} D_3 - 5 \frac{\partial}{\partial \tau_0} D_1 + D_2 \right) \tag{61}$$

where

$$D_1 = \int_1^{\tau_o} \frac{\tau^2\, d\tau}{\sqrt{\tau(1-\tau)(\tau-\tau_o)}} = -2\int_0^K dn^4\, u du = \frac{2}{3}\,\tau_o^2 K - \frac{4}{3}\,(1+\tau_o^2)E$$

$$D_2 = \int_1^{\tau_o} \frac{\tau\, d\tau}{\sqrt{\tau(1-\tau)(\tau-\tau_o)}} = -2\int_0^K dn^2\, u du = -2E$$

$$D_3 = \int_1^{\tau_o} \frac{\tau^3\, d\tau}{\sqrt{\tau(1-\tau)(\tau-\tau_o)}} = -2\int_0^K dn^6\, u du = \frac{3}{5}\,\tau_o\, D_2 + \frac{4}{5}\,(1+\tau_o)\, D_1$$

and the modulus of the elliptic integrals E and K is $\sqrt{1-\tau_o}$. Direct calculation gives for the coefficient of a_o the expression

$$\frac{E\,(3-\tau_o) - 2K}{(1-\tau_o)^2}$$

and for the coefficient of a_2

$$\frac{2\,\tau_o\,E - \tau_o\,K - \tau_o^2\,K}{(1-\tau_o)^2}$$

The value of b from equation (61) thus establishes for pressure coefficient the value

$$C_p = \frac{2x}{\beta^2(1-\theta_o^2)^2}\left\{ a_o\left[2K - E(3-\theta_o^2)\right] + a_2\left[(\theta_o^4+\theta_o^2)\,K - 2\theta_o^2\,E\right]\right\} \tag{62}$$

where the modulus of the elliptic integrals is $\sqrt{1-\theta_o^2}$.

From equation (55), the ordinate of the upper surface of the wing is

$$z_u = \int_{\frac{y}{m}}^{x} \frac{dz_u}{dx}\, dx = \frac{1}{\beta^2}\int_{\frac{y}{m}}^{x} \frac{a_o x^2 + a_2\beta^2 y^2}{\sqrt{m_o^2 x^2 - y^2}}\, dx$$

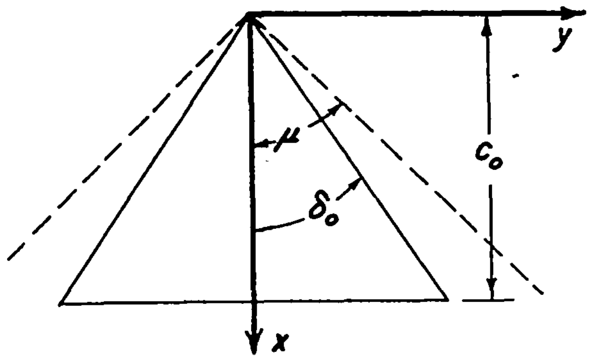
$$= \frac{a_o x}{2\beta^2 m_o^2}\sqrt{m_o^2 x^2 - y^2} + \frac{y^2}{2m_o^3\beta^2}\,(a_o + 2\beta^2 m_o^2 a_2)\ \text{arc cosh}\,\frac{m_o x}{y} \tag{63}$$

In reference 11, Squire considered the thickness distribution that is obtained by neglecting the arc hyperbolic function in equation (63). His results correspond to the case when a_o is $-2\beta^2 m_o^2 a_2$ and are specifically

$$z_u = \frac{a_o x}{2\beta^2 m_o^2} \sqrt{m_o^2 x^2 - y^2} \tag{64a}$$

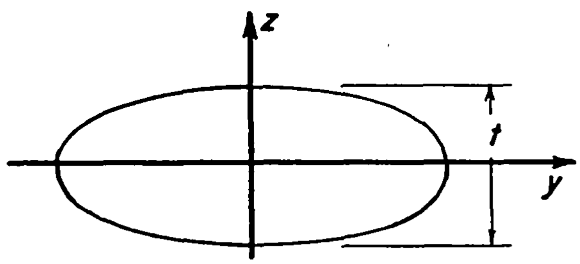
$$C_p = \frac{a_o x}{\beta^2 (1-\theta_o^2)^2} \left[(3-\theta_o^2)\, K - (4-2\theta_o^2)\, E \right] \tag{64b}$$

If the wing is cut normal to the stream direction to form a trailing edge, a triangular plan form and an elliptical cross-section result as shown in the accompanying sketch. If the root chord of the wing is c_o and the maximum thickness at the trailing edge is t, the constant a_o in equation (64a) is equal to $\beta^2 m_o t / c_o^2$ and the analytical expressions for the upper surface of the wing and pressure coefficient are



$$\frac{2z_u}{t} = \frac{x}{c_o} \sqrt{\left(\frac{x}{c_o}\right)^2 - \left(\frac{y}{m_o c_o}\right)^2} \tag{65a}$$

$$C_p = \frac{m_o t x}{c_o^2 (1-\theta_o^2)^2} \left[(3-\theta_o^2)K - (4-2\theta_o^2)E \right] \tag{65b}$$



It is apparent that a multiplicity of thickness distributions with the same pressure distributions must exist. Consider first the case when a_o in equations (62) and (63) is zero. The surface ordinate and pressure coefficient are, respectively,

$$z_u = \frac{a_2 y^2}{m_o} \operatorname{arc\,cosh} \frac{m_o x}{y} \tag{66a}$$

$$C_p = \frac{2a_2 x\,\theta_o^2}{\beta^2(1-\theta_o^2)^2}\left[(1+\theta_o^2)\,K - 2E\right] \tag{66b}$$

The lateral section of such a wing is shown for particular values of m_o and β in the next sketch (the curve denoted by $n = \infty$). Along the root chord the thickness is zero while the maximum thickness position occurs at the value of $m_o x/y$ satisfying equation

$$\frac{m_o x/y}{2\sqrt{(m_o x/y)^2-1}} = \text{arc cosh}\,\frac{m_o x}{y}$$

or at approximately

$$m_o x/y = 1.3128 \quad \text{or} \quad y/m_o x = 0.7617$$

The pressure coefficients of equations (65b) and (66b) are identical provided the equality

$$\frac{a_2}{\beta^2} = \frac{m_o t}{2\theta_o^2 c_o^2}\;\frac{[(3-\theta_o^2)\,K - (4-2\theta_o^2)E]}{[(1+\theta_o^2)\,K - 2E]}$$

holds. From equations (65a) and (66a) it follows that the surfaces

$$\frac{2Z_1}{t} = \left(\frac{x}{c_o}\right)^2\sqrt{1-\left(\frac{y}{m_o x}\right)^2} \tag{67a}$$

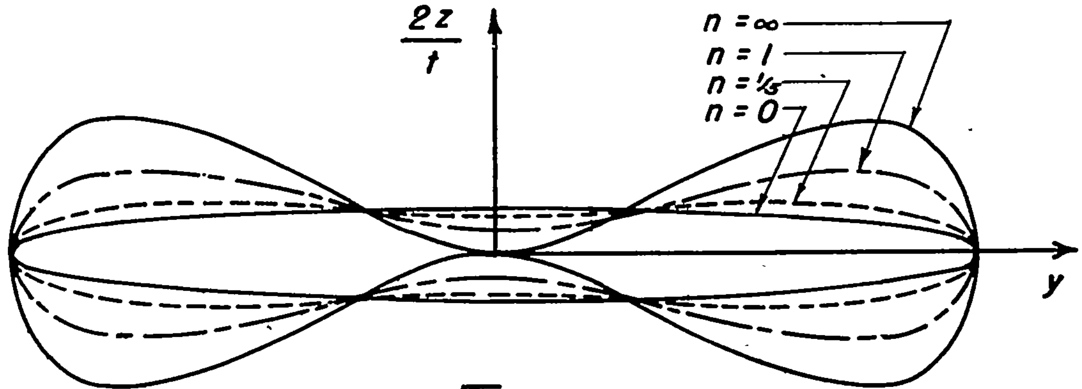
$$\frac{2Z_2}{t} = \left(\frac{y}{m_o c_o}\right)^2\frac{[(3-\theta_o^2)\,K - (4-2\theta_o^2)E]}{[(1+\theta_o^2)\,K - 2E]}\,\text{arc cosh}\,\frac{m_o x}{y} \tag{67b}$$

have precisely the same pressure distribution as does also any surface given by the relation

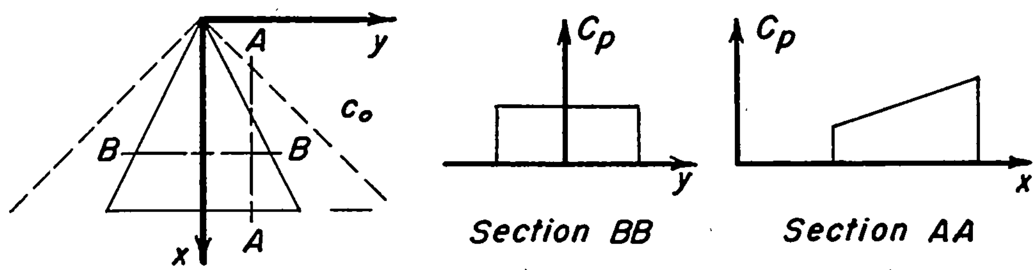
$$\frac{2Z}{t} = \frac{2Z_1 + 2nZ_2}{t(1+n)} \tag{67c}$$

where n is an arbitrary multiplicative factor.

Cross sections in the $x = c_o$ plane of surfaces given by equations (67a) and (67b) are shown in the accompanying sketch for



$x = c_o$, $M_o = \sqrt{2}$, $m_o = 1/2$



$M_o = \sqrt{2}$ and $m_o = 1/2$. Also included are sections calculated from equation (67c) for $n = 1$ and $n = 1/5$.

Since bodies with the same pressure distribution can be found, the possibility of combining results and getting a body inducing no change in the free-stream pressure should be investigated. From equation (62), it follows that for $C_p = 0$, the arbitrary constants a_o and a_2 must satisfy the relation

$$\frac{a_2}{a_o} = -\frac{2K - E\ (3-\theta_o^2)}{\theta_o^2[(1+\theta_o^2)\ K - 2E]}$$

and, from equation (63), the ordinate of the resulting surface is expressible in the form

$$\frac{2\beta^2 m_o^3}{a_o y^2} z_u = \left(\frac{m_o x}{y}\right)\sqrt{\left(\frac{m_o x}{y}\right)^2 - 1} - P_3 \text{ arc cosh } \frac{m_o x}{y}$$

where

$$P_3 = P_2/P_1$$

and

$$P_1 = (1+\theta_o^2)\,K - 2E$$

$$P_2 = (3-\theta_o^2)\,K - (4-2\theta_o^2)\,E$$

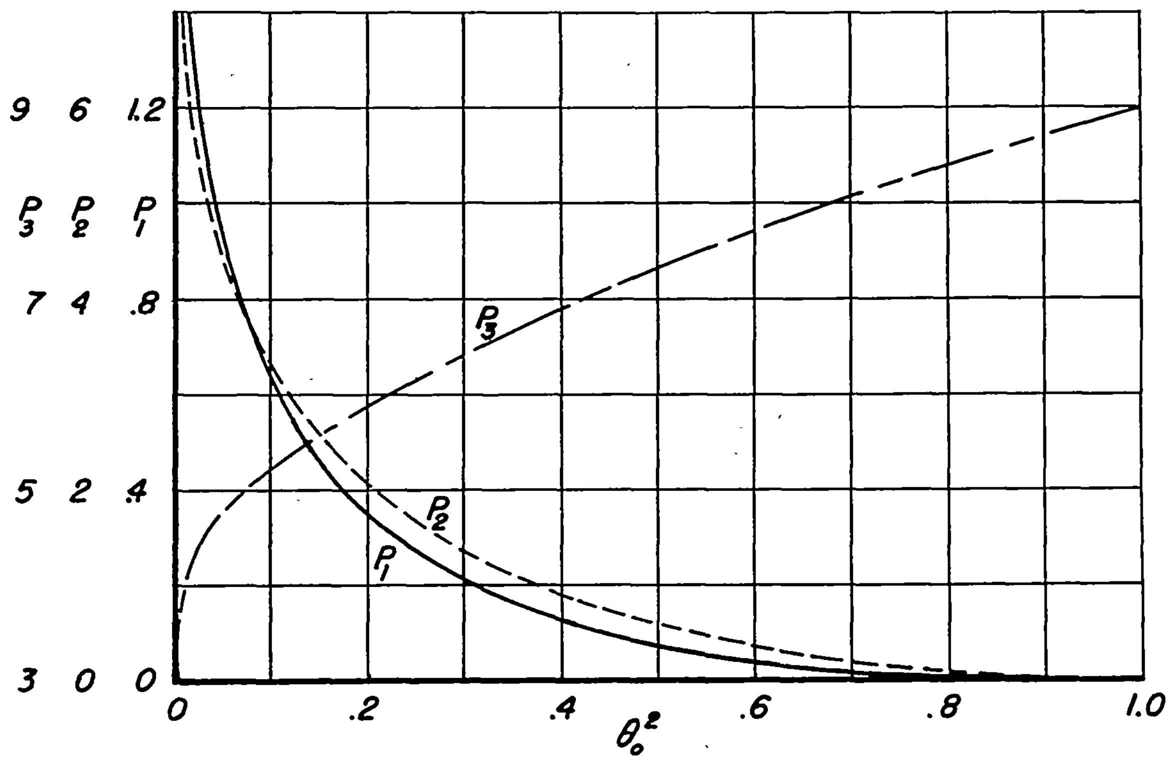
The surface produces a real wing only if z_u remains positive and this condition leads to the inequality

$$(m_o x/y)\,\sqrt{(m_o x/y)^2 - 1}\ \Big/\ \text{arc cosh}\ m_o x/y \geqq P_3$$

The range of $m_o x/y$ is from 1 to ∞ and it is easy to show that the left member has the lower limit 1 at $m_o x/y$ equal to 1. On the other hand, P_3 can be written in the form

$$P_3 = \frac{P_2}{P_2 - 2(1-\theta_o^2)(K - E)}$$

and, since $K - E > 0$ for $\theta_o < 1$, it appears likely that P_3 is greater than one. A more detailed check shows, in fact, that P_3 lies between 3 and 9 from which it follows that the inequality can never be satisfied in the neighborhood of the leading edge and no real wing with zero pressure coefficient is possible. The variation of P_1, P_2, and P_3 is shown in the accompanying sketch in which the variables are plotted as functions of θ_o^2.



CONCLUDING REMARKS

It has been shown that the assumption of quasi-conical flow in a supersonic field transforms the basic partial differential equation for the perturbation potential to an elliptic-type equation in two independent variables throughout the region inside the Mach cone. It is therefore not surprising that solutions of wing problems, for both the lifting and the nonlifting case, lead directly to the consideration of an integral equation (equation (18)) of the type encountered in two-dimensional subsonic theory. In the analysis and the applications of this report, it is shown that for a large class of specified conditions the known inversion of the integral equation produces solutions that require straightforward integrations and the solving of simultaneous linear equations.

An unusual feature of the resulting theory is the fact that a multiplicity of solutions may appear in a given problem. In retrospect, this degree of freedom is not surprising since it is well known that a null solution exists in two-dimensional subsonic theory and appears in lifting problems in the form of a purely circulatory flow. In the study of subsonic symmetrical profiles, this arbitrariness in the solution occurs when the geometry of the wing is to be determined from the distribution of pressure exerted by the fluid. Since, however, closure of the wing is necessary, an additional condition is given which establishes uniqueness in much the same manner that the Kutta condition imposes uniqueness in the lifting case. In the consideration of supersonic quasi-conical flow, similar conditions to determine uniqueness do not necessarily arise. Mathematically, the condition of uniqueness is determined from the degree of dependence between simultaneous linear equations.

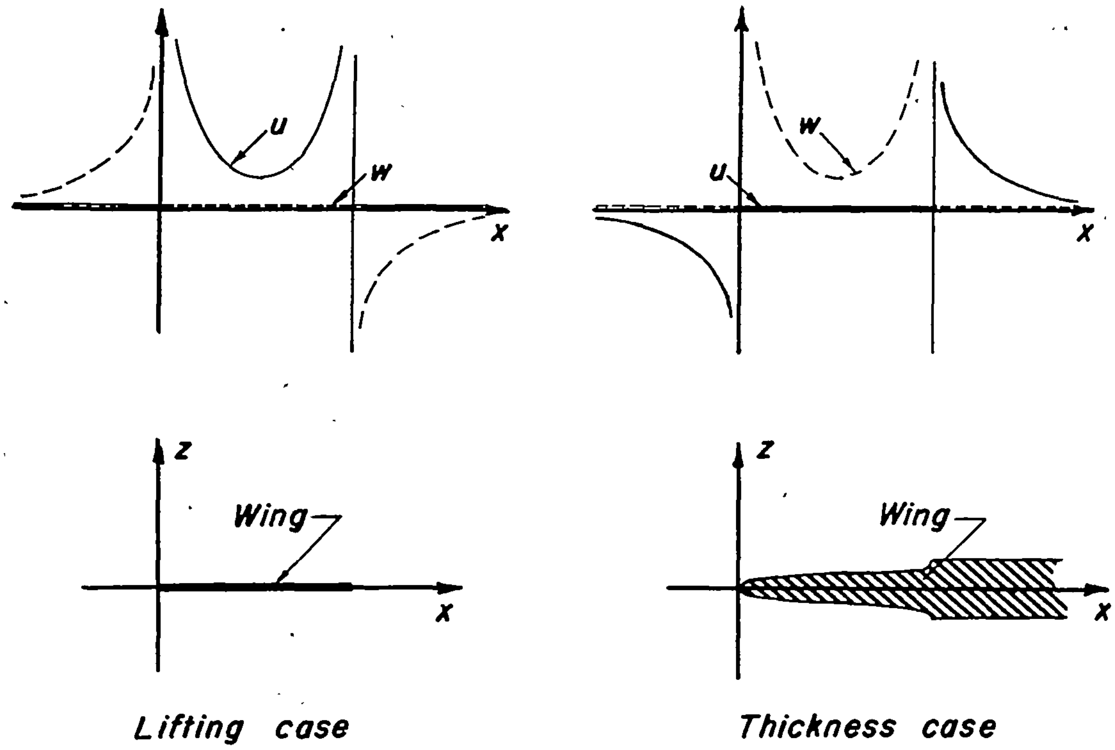
One further remark concerning the analogy between lifting and nonlifting problems appears to be pertinent. In two-dimensional subsonic theory the integral relation between the perturbation velocities along the wing surface may be written, in the lifting case, as

$$w(x,o) = \frac{-\beta}{\pi} \int_o^c \frac{u(x_1,o)dx_1}{x-x_1}$$

where c is chord length. In the symmetrical case, the interrelation is expressed in the form

$$u(x,o) = \frac{1}{\pi\beta} \int_o^c \frac{w(x_1,o)dx_1}{x-x_1}$$

Thus, aside from the factor β, a complete duality exists in the formal mathematical analysis of the two problems. Hence, the circulatory motion associated with a flat plate at zero angle of attack (see sketch) is



analogous to a slope distribution associated with zero pressure coefficient. However, in three-dimensional supersonic wing theory the basic relations between w and u on the wing do not have the property of duality (see equations (10) and (27)). The assumption of quasi-conical flow, however, together with the restriction to triangular-type plan forms with subsonic leading edges, brings the study of lift and thickness into more general parallelism and a close similarity exists between the final expressions in equations (25) and (32).

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., July 10, 1951

APPENDIX

THE GENERALIZED PRINCIPAL PART AND FINITE PART OF AN INTEGRAL

If the singular integrand of a convergent improper integral is differentiated formally, without due regard for the singularity, the resulting expression is, in general, improper. In applied theory, however, the differentiation is usually to be performed upon the integral itself and in this case a careful treatment of the entire expression leads to a finite answer. The two most common examples of such problems arise in the evaluation of the Cauchy principal part and Hadamard's finite part. The following development indicates the manner in which these cases are extended to include multiple differentiations. (The generalized principal part shall be concerned with integrands having singularities within the region of integration and of order n where n is a positive integer; the finite part, on the other hand, involves integrands with singularities at an end point of the region of integration and of order $n + 1/2$.)

Consider first the evaluation of Cauchy's principal part. In this case a single differentiation is used and the expression

$$g(x) = \frac{\partial}{\partial x} \int_a^b A(x_1)\ \ln |x-x_1|\ dx_1, \quad a<x<b \tag{A1}$$

becomes, for constants a and b,

$$g(x) = \mathop{\int\!\!\!\!\!\!-}_a^b \frac{A(x_1)dx_1}{x-x_1} \tag{A2}$$

Here the symbol on the integral sign indicates that $g(x)$ is to be evaluated by a limiting process defined as follows

$$g(x) = \lim_{\epsilon \to 0} \left[\int_a^{x-\epsilon} \frac{A(x_1)}{x-x_1}\, dx_1 + \int_{x+\epsilon}^b \frac{A(x_1)}{x-x_1}\, dx_1 \right] \tag{A3}$$

To assure the convergence of this integral it is sufficient but not necessary to assume that $A(x)$ is differentiable at the point $x_1 = x$ and that elsewhere within the region of integration $A(x_1)$ is either continuous or possesses integrable singularities. The concept of the Cauchy principal part is so well known that the symbol on the integral is often omitted, as shall be done here.

Turning next to the case of multiple differentiations, consider the expressions

$$I \equiv \frac{\partial}{\partial x} \int_a^b \frac{dx_1}{x_1-x} \equiv \int_a^b \frac{\partial}{\partial x} \frac{dx_1}{x_1-x} = \int_a^b \frac{dx_1}{(x-x_1)^2},\ a<x<b \tag{A4}$$

where the limits of integration are independent of x and the symbols on the two latter integrals indicate that the generalized principal part is to be calculated. From the definition of I in the first integral, it follows that

$$I = \frac{\partial}{\partial x} \ln \frac{b-x}{x-a} = -\frac{b-a}{(b-x)\ (x-a)} \tag{A5}$$

The simple definition given by equation (A4) can be generalized to include integrals of the type

$$\begin{aligned} I_2 &= \frac{-\partial^2}{\partial x^2} \int_a^b A(x_1)\ \ln |x-x_1|\ dx_1 = \frac{\partial}{\partial x} \int_a^b \frac{A(x_1)\ dx_1}{x_1-x} \\ &= \int_a^b \frac{A(x_1)\ dx_1}{(x_1-x)^2} \end{aligned} \tag{A6}$$

Equation (A6) defines the symbol appearing in the final member. It is possible, however, to relate this integration to the particular integrand in the final integral of equation (A4) by writing I_2 in the form

$$I_2 = \frac{\partial}{\partial x} \left[\int_a^b \frac{A(x_1) - A(x)}{x_1-x}\, dx_1 + A(x) \int_a^b \frac{dx_1}{x_1-x} \right]$$

Then if $A(x_1)$ is integrable and if, at $x_1 = x$, its derivative exists and is single valued, the expression for I_2 becomes

$$\begin{aligned} I_2 &= \int_a^b \frac{A(x_1) - A(x)}{(x_1-x)^2}\, dx_1 + A(x) \int_a^b \frac{dx_1}{(x_1-x)^2} \\ &= \int_a^b \frac{A(x_1) - A(x)}{(x_1-x)^2}\, dx_1 - \frac{A(x)\ (b-a)}{(b-x)\ (x-a)} \end{aligned} \tag{A7}$$

where the results of equation (A5) have been used.

The first integral in equation (A7) is now in a form that involves no extension beyond the concept of Cauchy's principal part and the evaluation of I_2 may be carried out with that form. Furthermore, it can be shown from equation (A7) that if the indefinite integral of $A(x_1)/(x_1-x)^2$ exists such that

$$\int \frac{A(x_1)dx_1}{(x_1-x)^2} = G(x_1,x) + C \tag{A8}$$

then the value of I_2 can be found by following the conventional rules for substitution of limits so that

$$⨍_a^b \frac{A(x_1)dx_1}{(x_1-x)^2} = G(b,x) - G(a,x) \tag{A9}$$

The extension to higher ordered derivatives is obvious. Thus, for a and b independent of x, one has

$$\begin{aligned} I_{n+1} &= -\frac{\partial^{n+1}}{\partial x^{n+1}} \int_a^b A(x_1)\, ln\, |x-x_1|\, dx_1 = \frac{\partial^n}{\partial x^n} \int_a^b \frac{A(x_1)dx_1}{(x_1-x)} \\ &= n!\, ⨍_a^b \frac{A(x_1)dx_1}{(x_1-x)^{n+1}} \end{aligned} \tag{A10}$$

Equation (A10) defines the final member appearing in it. The quantity I_{n+1} can also be written in the form

$$I_{n+1} = n! \left[\int_a^b \frac{A(x_1) - B(x,x_1)}{(x_1-x)^{n+1}}\, dx_1 + ⨍_a^b \frac{B(x,x_1)dx_1}{(x_1-x)^{n+1}} \right] \tag{A11}$$

where

$$B(x,x_1) = A(x) + \frac{A'(x)}{1}(x_1-x) + \ldots + \frac{A^{(n-1)}(x)}{(n-1)!}(x_1-x)^{n-1}$$

and

$$⨍_a^b \frac{dx_1}{(x_1-x)^{i+1}} = \frac{1}{i!}\left(\frac{\partial}{\partial x}\right)^i \int_a^b \frac{dx_1}{x_1-x} = -\frac{1}{i}\left[\frac{1}{(b-x)^i} + (-1)^{i-1}\frac{1}{(x-a)^i}\right], i \leq n$$

The first n derivatives of $A(x_1)$ are assumed to exist and be single valued at $x_1 = x$ while elsewhere in the range of integration $A(x)$ may possess integrable singularities. The generalization of equation (A9) holds so that if

$$\int \frac{A(x_1)\,dx_1}{(x_1-x)^{n+1}} = G(x_1,x) + C \tag{A12}$$

then

$$⨎_a^b \frac{A(x_1)\,dx_1}{(x_1-x)^{n+1}} = G(b,x) - G(a,x) \tag{A13}$$

It is also possible to extend the definition of equation (A10) to include a functional dependency on x in the numerator of the integrands. Thus, replacing $A(x_1)$ by $A(x,x_1)$, equation (A10) again defines uniquely a principal part integral provided the first n derivatives of $A(x,x_1)$ with respect to x and x_1 exist at $x_1 = x$.

The original concept of the finite part was used by Hadamard in connection with square root singularities. Consider the expressions

$$J \equiv \frac{\partial}{\partial b} \int_a^b \frac{dx_1}{\sqrt{b-x_1}} \equiv ⨍_a^b \frac{\partial}{\partial b} \frac{dx_1}{\sqrt{b-x_1}} = -\frac{1}{2} ⨍_a^b \frac{dx_1}{(b-x_1)^{3/2}} \tag{A14}$$

From the first integral in this relation it follows that

$$⨍_a^b \frac{dx_1}{(b-x_1)^{3/2}} = \frac{-2}{\sqrt{b-a}} \tag{A15}$$

The natural extension of this idea is to consider

$$J_2 \equiv \frac{\partial}{\partial b} \int_a^b \frac{A(x_1)dx_1}{\sqrt{b-x_1}} \equiv ⨍_a^b \frac{\partial}{\partial b} \frac{A(x_1)dx_1}{\sqrt{b-x_1}} = -\frac{1}{2} ⨍_a^b \frac{A(x_1)dx_1}{(b-x_1)^{3/2}} \tag{A16}$$

where $A(x_1)$ is continuous at $x_1 = b$ and is integrable elsewhere in the range of integration.

The evaluation of J_2 can be related more closely to the integral of equation (A14) after rewriting J_2 in the form

$$J_2 = \frac{\partial}{\partial b}\left[\int_a^b \frac{A(x_1) - A(b)}{\sqrt{b-x_1}}\,dx_1 + A(b) \int_a^b \frac{dx_1}{\sqrt{b-x_1}}\right] \tag{A17}$$

It follows that

$$\int_a^b \frac{A(x_1)dx_1}{(b-x_1)^{3/2}} = \int_a^b \frac{A(x_1) - A(b)}{(b-x_1)^{3/2}}\,dx_1 + A(b) \int_a^b \frac{dx_1}{(b-x_1)^{3/2}} \tag{A18}$$

An interesting integration technique can be evolved from equation (A18). Setting

$$\int_a^b \frac{A(x_1)dx_1}{(b-x_1)^{3/2}} = \lim_{\beta \to b} \int_a^\beta \frac{A(x_1) - A(b)}{(b-x_1)^{3/2}}\,dx_1 - \frac{2A(b)}{\sqrt{b-a}}$$

and setting the indefinite integral of

$$\int \frac{A(x_1)dx_1}{(b-x_1)^{3/2}}$$

equal to $F(b,x_1) + C$, it follows that

$$\int_a^b \frac{A(x_1)dx_1}{(b-x_1)^{3/2}} = -\left[F(b,a) + C\right]$$

where

$$C = \lim_{x_1 \to b}\left[\frac{2A(b)}{\sqrt{b-x_1}} - F(b,x_1)\right]$$

Thus, with the proper choice of the constant of integration, the definite integral is found by substituting conventionally the lower limit. In practice, C is often zero.

Defining J_{n+1} in the form

$$J_{n+1} = \left(\frac{\partial}{\partial b}\right)^n \int_a^b \frac{A(x_1)dx_1}{\sqrt{b-x_1}} = \int_a^b \left(\frac{\partial}{\partial b}\right)^n \frac{A(x_1)}{\sqrt{b-x_1}}\,dx_1$$

$$= (-1)^n \frac{1.3\ldots(2n-1)}{2^n} \int_a^b \frac{A(x_1)dx_1}{(b-x_1)^{n+1/2}} \tag{A19}$$

it follows that

$$\int_a^b \frac{A(x_1)dx_1}{(b-x_1)^{n+\frac{1}{2}}} = \int_a^b \frac{A(x_1) - B(b,x_1)}{(b-x_1)^{n+\frac{1}{2}}} dx_1 + \int_a^b \frac{B(b,x_1)dx_1}{(b-x_1)^{n+\frac{1}{2}}} \qquad \text{(A20)}$$

where

$$B(b,x_1) = A(b) - A'(b)(b-x_1) + \ldots + \frac{(-1)^{n-1}A^{(n-1)}(b)}{(n-1)!}(b-x_1)^{n-1}$$

and

$$\int_a^b \frac{dx_1}{(b-x_1)^{i+\frac{1}{2}}} = \frac{(-1)^i 2^i}{1.3\ldots(2i-1)} \left(\frac{\partial}{\partial b}\right)^i \int_a^b \frac{dx_1}{(b-x_1)^{\frac{1}{2}}} = -\frac{2}{2i-1} \frac{1}{(b-a)^{i-\frac{1}{2}}}$$

It is furthermore possible to extend the definition of equation (A19) to include a functional dependence on b of the integrand $A(x_1)$. Replacing $A(x_1)$ by $A(b,x_1)$, equation (A19) again defines uniquely a finite part integral provided that

$$\lim_{x_1 \to b} \left[\frac{\partial^n A(b,x_1)}{\partial b^n} \sqrt{b-x_1} \right] = 0$$

REFERENCES

1. Busemann, Adolf: Infinitesimale keglige Uberschallstromung. Deutsche Akademie der Luftfahrtforschung, Schriften, vol. 7B, no. 3, 1943, pp. 105-122. (Available as NACA TM 1100)

2. Brown, Clinton E. and Adams, Mac C.: Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds. NACA Rep. 892, 1948. (Formerly NACA TN 1566)

3. Ribner, Herbert S.: Some Conical and Quasi-Conical Flows in Linearized Supersonic-Wing Theory. NACA TN 2147, 1950.

4. Lagerstrom, P.A.: Linearized Supersonic Theory of Conical Wings. NACA TN 1685, 1948.

5. Germain, Paul: The General Theory of Conical Motions And Its Application to Supersonic Aerodynamics. Office National d'Etudes et de Recherches Aeronautique, Pub. No. 34, 1949.

6. Brown, Clinton E.: Theoretical Lift and Drag of Thin Triangular Wings at Supersonic Speeds. NACA Rep. 839, 1946. (Formerly NACA TN 1183)

7. Heaslet, Max. A., Lomax, Harvard, and Jones, Arthur L.: Volterra's Solution of the Wave Equation as Applied to Three-Dimensional Supersonic Airfoil Problems. NACA Rep. 889, 1947. (Formerly NACA TN 1412)

8. Heaslet, Max. A. and Lomax, Harvard: The Application of Green's Theorem to the Solution of Boundary-Value Problems in Linearized Supersonic Wing Theory. NACA Rep. 961, 1950. (Formerly NACA TN 1767)

9. Hadamard, Jacques: Lectures on Cauchy's Problem in Linear Partial Differential Equations. Yale University Press, New Haven, Conn., 1923.

10. Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B.: Formulas for Source, Doublet, and Vortex Distributions in Supersonic Wing Theory. NACA TN 2252, 1950.

11. Squire, H. B.: Theory of the Flow Over a Particular Wing in a Supersonic Stream. Rep. No. Aero 2184, British, 1947.

NACA-Langley - 9-19-51 - 1000